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ON THE EXACT DISTRIBUTION OF THE MAXIMUM OF THE EXPONENTIAL OF THE GENERALIZED NORMAL-INVERSE GAUSSIAN PROCESS WITH RESPECT TO A MARTINGALE MEASURE

ROMAN V. IVANOV

Abstract. In this paper we obtain explicit formulas for distributions of extrema of exponentials of time-changed Brownian motions with drift which generalize normal inverse Gaussian processes. The generalization is made by multiplying the normal inverse Gaussian process by a constant. The results are established with respect to the equivalent martingale measure. As examples of applications, problems of path-dependent option pricing are discussed.

1. Introduction

Firstly, a class of normal-inverse Gaussian (NIG) distributions was introduced in the work [1]. The definition of this class of distributions was induced by the reason of necessity of a better description of some phenomenons in geology. Together with its applications to geology and the theory of turbulence, the NIG distribution, as it was shown in the papers [2] , [3], [14], [15], fitted well as a model of the distribution of logarithmic asset returns. These facts measured interest to investigations of properties of the NIG distribution and NIG process. In the work [4], the analysis of the NIG process is given, including the Lévy measure of the NIG process deriving, obtaining properties of the NIG process, proposing an Ornstein-Uhlenbeck process of the NIG type and NIG type stochastic volatility model. For basic properties of the NIG process, see also the monograph [17]. In the brochure [13], some approximations of the NIG process are discussed. In the work [5], many numerical procedures which are used for computing asset prices, in particular, in the NIG models, are considered.

Besides the Brownian motion, semi-analytical results on the distribution of the extrema of a Lévy process are known for the processes with hyper-exponential jumps, with jumps of rational transform, meromorphic processes, if the supremum is calculated at an exponentially distributed time, see for details the works [10] and [11]. Applications of these results to option pricing are discussed in the paper [16]. For methods of the numerical calculation of the distributions of the extrema of time-changed Brownian motions, see the paper [7]. However, some explicit results

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on the NIG process are available: in optimal stopping, see the work [8], and in option pricing, see the paper [9].

In this paper we obtain an explicit formula for the distributions of the extrema of a simpler extension of the exponential of the NIG process discussing the NIG process as a time-changed Brownian motion. The distributions are calculated at deterministic times. The established formulas are given under the equivalent martingale measure. The author wishes that the derived results can be useful in mathematical finance, for example, for risk-neutral path-dependent option pricing.

2. Setup and the Distribution Function

Assume that a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is given. Let us generalize the NIG process as a time-changed Brownian motion

$$H_t^{gen} = \beta \tilde{T}(t) + \sigma B_{\tilde{T}(t)}, \tag{2.1}$$

where $B = (B_s)_{s \geq 0}$ is a P-Brownian motion, $\sigma > 0$, and the changed time is defined as

$$\tilde{T}(t) = \inf\{s \geq 0 : \tilde{B}_s + \tilde{a}s \geq \delta t\} \tag{2.2}$$

under parameters

$$\delta > 0, \quad \alpha > 0, \quad \tilde{a}^2 = \alpha^2 - \left(\frac{\beta}{\sigma}\right)^2 \geq 0.$$

It is supposed in (2.2) that $(\tilde{B}_s)_{s \geq 0}$ is a Brownian motion independent from the Brownian motion B. Notice that if $\sigma = 1$, this process becomes the normal-inverse Gaussian process $H = (H_t)_{t \geq 0}$ with the density function

$$f(H_t, x) = \frac{\alpha \delta t K_1\left(\alpha\sqrt{(\delta t)^2 + x^2}\right)}{\pi\sqrt{(\delta t)^2 + x^2}} e^{a\delta t + \beta x},$$

where $K_\gamma(x)$ is the MacDonald function (the modified Bessel function of the second kind) of index γ, and the Lévy measure

$$\lambda_H(x)dx = \frac{\delta\alpha}{\pi|x|} e^{\beta x} K_1(\alpha|x|)dx.$$

We set

$$\kappa = -\frac{1}{2} - \frac{\beta}{\sigma^2},$$

and define a new probability measure P^* equivalent to the measure P by the density

$$\frac{d\mathrm{P}_t^*}{d\mathrm{P}_t} = \frac{\exp(\kappa H_t^{gen})}{\mathrm{E}\exp(\kappa H_t^{gen})}. \tag{2.3}$$

It is not difficult to notice that the process

$$S_t = \exp(H_t^{gen}) \tag{2.4}$$

is a martingale with respect to P^*. Indeed, P^* is the Esscher measure, and with respect to the measure P^* the NIG process $\sigma^{-1}H^{gen} = (\sigma^{-1}H_t^{gen})_{t\geq 0}$ has the Lévy measure

$$\lambda^*_{\sigma^{-1}H^{gen}}(x)dx = e^{\kappa\sigma x}\frac{\delta\alpha}{\pi|x|}e^{\frac{\beta}{\sigma}x}K_1(\alpha|x|)dx, \tag{2.5}$$

see for details Theorem 2 in Chapter VII.3c of [17] on the change of time, and the brochure [13] for the NIG process case.

Therefore,

$$\mathit{Law}\,(H_t^{gen};\mathrm{P}^*) = \mathit{Law}\left(\sigma B_{T^*(t)} - \frac{\sigma^2}{2}T^*(t);\mathrm{P}\right),$$

where

$$T^*(t) = \inf\{s\geq 0:\ B_s^* + a^*s \geq \delta t\},\quad a^* = \sqrt{\alpha^2 - \frac{\sigma^2}{4}}$$

and $B^* = (B_t^*)_{t\geq 0}$ is an independent Brownian motion.

Throughout this work, we are interested in probabilities

$$\mathrm{P}^*\left(\overline{S}_t \leq u\right),\qquad u > S_0 = 1,$$

where

$$\overline{S}_t = \max_{u\leq t} S_u,\qquad t\geq 0.$$

To formulate our results, we need to introduce some special mathematical functions. Namely, the degenerate Appell hypergeometric function, or the degenerate generalized hypergeometric function of two variables.

For $\alpha\in\mathbb{R}$, set

$$(\alpha)_0 = 1,\ (\alpha)_m = \alpha(\alpha+1)...(\alpha+m-1),\ m = 1,2,...$$

Due to [18], there are four kinds of the Appell hypergeometric functions. These are formal extensions of hypergeometric functions to two variables which were introduced by P. Appell in 1880. Throughout this paper, we will deal with one of the four kinds which is

$$\mathrm{F}_2(\alpha,\beta,\beta',\gamma;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}}x^m y^n.$$

This sum is absolutely convergent if $|x|<1$ and $|y|<1$. Its β'-degenerate Appell function (in this paper, we call it simply the degenerate Appell function) is

$$\Phi(\alpha,\beta,\gamma;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{m!n!(\gamma)_{m+n}}x^m y^n,\qquad |x|<1,\ |y|<1.$$

Let us notice that derived formulas for the extrema have to be understood as approximate, i.e. as upper boundaries.

Finally, let us introduce the condition

$$\log u + \sqrt{\log^2 u + (\sigma\delta t)^2} < 2 \tag{2.6}$$

for the parameters.

Theorem 2.1. *Assume that $S = (S_t)_{t\geq 0}$ is the exponential of the generalized NIG process (2.4), where $H^{gen} = (H_t^{gen})_{t\geq 0}$ with $\sigma = 2\alpha$ is defined in (2.1). Let the equivalent martingale measure* P^* *be defined by (2.3) and $u > 1$, and let the condition (2.6) hold. Then for $t > 0$*

$$\begin{aligned}&\mathrm{P}^*\left(\overline{S}_t \leq u\right)\\ &= \frac{de^d}{\pi\sqrt{2}}\left\{\mathrm{B}\left(\frac{1}{2},1\right)(K_1(d)-K_0(d))\left[(1+q)^{\frac{1}{2}}\Phi_+^1-(1-q)^{\frac{1}{2}}e^{-\log u}\Phi_-^1\right]\right.\\ &\quad\left.+\,\mathrm{B}\left(\frac{3}{2},1\right)K_0(d)\left[(1+q)^{\frac{3}{2}}\Phi_+^2-(1-q)^{\frac{3}{2}}e^{-\log u}\Phi_-^2\right]\right\},\end{aligned}\tag{2.7}$$

where $\mathrm{B}(x,y)$ *is the beta function,* $K_\gamma(x)$ *is the MacDonald function of index* γ,

$$q = \frac{\log u}{\sqrt{\log^2 u + (\sigma\delta t)^2}} \qquad \textit{and} \qquad d = \frac{1}{2}\sqrt{\log^2 u + (\sigma\delta t)^2},$$

and for $\iota \in \{+,-\}$ Φ_ι^1 and Φ_ι^2 are values of the degenerate Appell functions

$$\Phi_\iota^1 = \Phi\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{1+\iota q}{2},-d(1+\iota q)\right)$$

and

$$\Phi_\iota^2 = \Phi\left(\frac{3}{2},\frac{1}{2},\frac{5}{2};\frac{1+\iota q}{2},-d(1+\iota q)\right).$$

A proof of Theorem 2.1 is given in section Proofs.

Let

$$\underline{S}_t = \min_{u\leq t} S_u, \qquad t \geq 0.$$

Then for $u < 1$

$$\begin{aligned}\mathrm{P}^*\left(\underline{S}_t \leq u\right) =&\mathrm{P}^*\left(\max_{v\leq t}(-H_v^{gen}) \geq -\log u\right)\\ =&\mathrm{P}^*\left(\max_{v\leq t}\left(e^{-H_v^{gen}}\right) \geq \frac{1}{u}\right),\end{aligned}\tag{2.8}$$

and the next corollary follows.

Corollary 2.2. *Let $u < 1$. Using the notations and conditions of Theorem 2.1*

$$\begin{aligned}&\mathrm{P}^*\left(\underline{S}_t \leq u\right) = e^{\log u}\\ &+\frac{de^d}{\pi\sqrt{2}}\left\{\mathrm{B}\left(\frac{1}{2},1\right)(K_1(d)-K_0(d))\left[(1+q)^{\frac{1}{2}}\Phi_+^1-(1-q)^{\frac{1}{2}}e^{\log u}\Phi_-^1\right]\right.\\ &\quad\left.+\,\mathrm{B}\left(\frac{3}{2},1\right)K_0(d)\left[(1+q)^{\frac{3}{2}}\Phi_+^2-(1-q)^{\frac{3}{2}}e^{\log u}\Phi_-^2\right]\right\}.\end{aligned}\tag{2.9}$$

A proof of Corollary 2.2 is included in section Proofs.

Example 2.3. The down-and-in cash option has the payoff at the expiry

$$\mathrm{BC}_T = K I_{\{\overline{S}_T \le u\}},$$

where K is the prespecified amount of cash. Assume that the rate of the bank account $r = 0$. Then the risk-neutral price (for details of the general pricing theory, we refer to the monographs [12], [17] and the brochure [13]) BC of the up-and-in cash barrier option is

$$\mathrm{BC} = K\mathrm{P}^* \left(\overline{S}_T \le u\right)$$

and is determined by the result of Theorem 2.1.

Example 2.4. The down-and-in asset option has the payoff

$$\mathrm{BA}_T = S_T I_{\{\overline{S}_T \le u\}}.$$

Recall (see (2.5)) that the density $\lambda^*_{\sigma^{-1}H^{gen}}(x)$ of the Lévy measure of $\sigma^{-1}H^{gen}$ with respect to measure P^* is

$$\lambda^*_{\sigma^{-1}H^{gen}}(x) = \frac{\delta\alpha}{\pi|x|} e^{-\frac{\sigma}{2}x} K_1(\alpha|x|).$$

Let us define a new measure P' by the density process

$$\frac{d\mathrm{P}'_t}{d\mathrm{P}^*_t} = S_t, \qquad t \le T. \tag{2.10}$$

Then the price BA of the down-and-in asset barrier option is

$$\mathrm{BA} = \mathrm{E}^* S_T I_{\{\overline{S}_T \le u\}} = \mathrm{P}' \left(\overline{S}_T \le u\right),$$

and the density $\lambda'_{\sigma^{-1}H^{gen}}(x)$ of the Lévy measure of $\sigma^{-1}H^{gen}$ with respect to the measure P' is

$$\lambda'_{\sigma^{-1}H^{gen}}(x) = \frac{\delta\alpha}{\pi|x|} e^{\frac{\sigma}{2}x} K_1(\alpha|x|).$$

Hence

$$Law\left(H^{gen}; \mathrm{P}'\right) = Law\left(H^{gen'}; \mathrm{P}\right),$$

where

$$H^{gen'}_t = \sigma B_{T'(t)} + \frac{\sigma^2}{2} T'(t) = \sigma B_{T^*(t)} + \frac{\sigma^2}{2} T^*(t), \tag{2.11}$$

and

$$\begin{aligned}\mathrm{BA} &= \mathrm{P}\left(\overline{H}^{gen'}_T \le \log u\right) = \mathrm{P}\left(\min_{t\le T}\left(\sigma B_{T^*(t)} - \frac{\sigma^2}{2} T^*(t)\right) \ge -\log u\right)\\ &= \mathrm{P}^*\left(\underline{S}_T \ge \frac{1}{u}\right).\end{aligned}$$

Therefore, the price BA of the down-and-in asset barrier option is determined by the result of Corollary 2.2.

3. Proofs

Proof of Theorem 2.1. Since

$$\begin{aligned}
\mathrm{P}^*\left(\overline{S}_t^{gen} \le u\right) &= \mathrm{P}^*\left(\max_{s\le t}\left(\beta\tilde{T}(s) + \sigma B_{\tilde{T}(s)}\right) \le \log u\right) \\
&= \mathrm{P}\left(\max_{s\le t}\left(-\frac{\sigma^2}{2}T^*(s) + \sigma B_{T^*(s)}\right) \le \log u\right)
\end{aligned}$$

with

$$T^*(t) = \inf\{s \ge 0 : \ B_s^* \ge \delta t\},$$

we have that

$$\mathrm{P}^*\left(\overline{S}_t^{gen} \le u\right) = \int_0^\infty \mathrm{P}\left(\max_{s\le g}\left(-\frac{\sigma^2}{2}s + \sigma B_s\right) \le \log u\right) f(T^*(t), g)dg, \tag{3.1}$$

where

$$f(T^*(t), g) = \frac{\delta t}{\sqrt{2\pi g^3}}\exp\left(-\frac{(\delta t)^2}{2g}\right). \tag{3.2}$$

Since

$$\begin{aligned}
&\mathrm{P}\left(\max_{s\le g}\left(-\frac{\sigma^2}{2}s + \sigma B_s\right) \le \log u\right) \\
&= \Psi\left(\frac{\log u}{\sigma\sqrt{g}} + \frac{\sigma}{2}\sqrt{g}\right) - e^{-\log u}\Psi\left(-\frac{\log u}{\sigma\sqrt{g}} + \frac{\sigma}{2}\sqrt{g}\right),
\end{aligned} \tag{3.3}$$

where $\Psi(x)$ is the distribution function of the standard normal random variable $N(0,1)$, it is enough to calculate the integral

$$I = \int_0^\infty \Psi\left(\frac{a}{\sqrt{g}} + b\sqrt{g}\right) g^{-\frac{3}{2}} e^{-\frac{c}{g}} dg$$

for

$$a \ne 0, \qquad b > 0, \qquad c > 0$$

under the condition

$$b\left(a + \sqrt{a^2 + 2c}\right) < 1 \tag{3.4}$$

which follows from (2.6).

To compute of I, let us use the extra parametrization method setting $a = a(v)$, $b = b(v)$, where $v \in (-\infty, V]$ and $a(V) = a$, $b(V) = b$. Then

$$\begin{aligned}
I &= \int_0^\infty \left(\int_{-\infty}^V \Psi'_v\left(\frac{a(v)}{\sqrt{y}} + b(v)\sqrt{y}\right) dv\right) y^{-\frac{3}{2}}\exp\left(-\frac{c}{y}\right) dy \\
&= \int_0^\infty \left(\int_{-\infty}^V \left(\Psi'_a\left(\frac{a}{\sqrt{y}} + b\sqrt{y}\right) a'_v + \Psi'_b\left(\frac{a}{\sqrt{y}} + b\sqrt{y}\right) b'_v\right) dv\right) \\
&\quad \times y^{-\frac{3}{2}}\exp\left(-\frac{c}{y}\right) dy.
\end{aligned}$$

Further, using the notations

$$I^1 = \int_0^\infty \Psi'_a \left(\frac{a}{\sqrt{y}} + b\sqrt{y} \right) y^{-\frac{3}{2}} \exp\left(-\frac{c}{y} \right) dy$$

and

$$I^2 = \int_0^\infty \Psi'_b \left(\frac{a}{\sqrt{y}} + b\sqrt{y} \right) y^{-\frac{3}{2}} \exp\left(-\frac{c}{y} \right) dy,$$

we get that

$$I = \int_{-\infty}^V (I^1(v)a_v + I^2(v)b_v)dv,$$

where $a_v = a'(v)$, $b_v = b'(v)$.

Next, the author uses the formula 3.471.9 from [6]. With respect to it,

$$\int_0^\infty x^{\alpha-1} e^{-\frac{\beta}{x} - \eta x} dx = 2 \left(\frac{\beta}{\eta} \right)^{\frac{\alpha}{2}} K_\alpha \left(2\sqrt{\beta\eta} \right), \tag{3.5}$$

if $\beta > 0$, $\eta > 0$. In (3.5), $K_\gamma(x)$ is the MacDonald function with index γ.

Using (3.5), we establish that

$$\begin{aligned} I^1 &= \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(a+by)^2}{2y} - \frac{c}{y} \right) y^{-2} dy \\ &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{a^2+2c}{2y} - \frac{b^2 y}{2} \right) y^{-2} dy \\ &= e^{-ab} \sqrt{\frac{2}{\pi}} \left(\frac{a^2+2c}{b^2} \right)^{-\frac{1}{2}} K_1 \left(|b| \sqrt{a^2+2c} \right) \end{aligned}$$

since $K_{-1}(x) = K_1(x)$, and

$$\begin{aligned} I^2 &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{a^2+2c}{2y} - \frac{b^2 y}{2} \right) y^{-1} dy \\ &= e^{-ab} \sqrt{\frac{2}{\pi}} K_0 \left(b\sqrt{a^2+2c} \right). \end{aligned}$$

Next, we imply for the simplicity of notations that

$$K_0 = K_0 \left(b\sqrt{a^2+2c} \right) \qquad \text{and} \qquad K_1 = K_1 \left(b\sqrt{a^2+2c} \right).$$

To simplify the integration of $I^1(v)$ and $I^2(v)$, let the author select a path of the integration in $\{(v,a,b)\}$-space such that the arguments of the MacDonald functions are constants at this path. Namely, set for $-\infty < v \le V = a$

$$a(v) = v$$

and

$$b(v) = \frac{b\sqrt{a^2+2c}}{\sqrt{v^2+2c}}.$$

Then

$$\int_{-\infty}^{a} I^1(v)a_v dv = \sqrt{\frac{2}{\pi}}K_1 \int_{-\infty}^{a} \exp\left(-\frac{vb\sqrt{a^2+2c}}{\sqrt{v^2+2c}}\right)\left(\frac{(v^2+2c)^2}{b^2(a^2+2c)}\right)^{-\frac{1}{2}} dv \quad (3.6)$$

and

$$\int_{-\infty}^{a} I^2(v)b_v dv = -\sqrt{\frac{2}{\pi}}K_0 \int_{-\infty}^{a} \exp\left(-\frac{vb\sqrt{a^2+2c}}{\sqrt{v^2+2c}}\right)\frac{vb\sqrt{a^2+2c}}{(v^2+2c)^{3/2}}dv. \quad (3.7)$$

Next, let us make a change of variables $v \to u$ in the integrals (3.6) and (3.7), $u = v/\sqrt{v^2+2c}$. Set $d = b\sqrt{a^2+2c}$. Then

$$v^2 = \frac{2cu^2}{1-u^2}, \qquad dv = \frac{\sqrt{2c}}{(1-u^2)^{\frac{3}{2}}}du,$$

and we get that

$$\int_{-\infty}^{a} I^1(v)a_v dv = \sqrt{\frac{2}{\pi}}K_1 \int_{-\infty}^{a} e^{-du}\left(\frac{v^2+2c}{d}\right)^{-1} dv$$
$$= \frac{1}{\sqrt{\pi c}}K_1 d \int_{-1}^{\frac{a}{\sqrt{a^2+2c}}} e^{-du}(1-u^2)^{-\frac{1}{2}}du$$

and

$$\int_{-\infty}^{a} I^2(v)b_v dv = \sqrt{\frac{2}{\pi}}K_0 \int_{-\infty}^{a} e^{-du}\left(\frac{v^2+2c}{d}\right)^{-1} udv$$
$$= \frac{1}{\sqrt{\pi c}}K_0 d \int_{-1}^{\frac{a}{\sqrt{a^2+2c}}} e^{-du}u(1-u^2)^{-\frac{1}{2}}du.$$

Further, set $q = |a|/\sqrt{a^2+2c}$, $\iota = \operatorname{sign}(a)$ and make a change of variables $u \to x$, $x = (1+u)/(1+\iota q)$. Then

$$\int_{-\infty}^{a} I^1(v)a_v dv = \frac{1}{\sqrt{\pi c}}K_1 d$$
$$\times \int_0^1 e^{-d(x(1+\iota q)-1)}\left(1-(x(1+\iota q)-1)^2\right)^{-\frac{1}{2}}(1+\iota q)dx$$
$$= \sqrt{\frac{1+\iota q}{2\pi c}}K_1 d e^d \int_0^1 e^{-d(1+\iota q)x}x^{-\frac{1}{2}}\left(1-\frac{1+\iota q}{2}x\right)^{-\frac{1}{2}} dx$$

and

$$\int_{-\infty}^{a} I^2(v)b_v dv$$
$$= (1+\iota q)\sqrt{\frac{1+\iota q}{2\pi c}}K_0 d e^d \int_0^1 e^{-d(1+\iota q)x}x^{\frac{1}{2}}\left(1-\frac{1+\iota q}{2}x\right)^{-\frac{1}{2}} dx$$
$$- \sqrt{\frac{1+\iota q}{2\pi c}}K_0 d e^d \int_0^1 e^{-d(1+\iota q)x}x^{-\frac{1}{2}}\left(1-\frac{1+\iota q}{2}x\right)^{-\frac{1}{2}} dx.$$

With respect to the formula 3.385 in [6],

$$\int_0^1 x^{\eta-1}(1-x)^{\lambda-1}(1-\beta x)^{-\rho}e^{-\mu x}dx = \mathrm{B}(\eta,\lambda)\Phi(\eta,\rho,\lambda+\eta;\beta,-\mu) \tag{3.8}$$

if $\lambda > 0$, $\eta > 0$, $|\beta| < 1$ and $|\mu| < 1$, where $\mathrm{B}(x,y)$ is the beta function and $\Phi(\alpha,\beta,\gamma;x,y)$ is the degenerate generalized hypergeometric function of two variables. Since condition (3.4) holds, we obtain from (3.8) that

$$\begin{aligned} I &= \int_{-\infty}^{a} I^1(v)a_v dv + \int_{-\infty}^{a} I^2(v)b_v dv \\ &= de^d\sqrt{\frac{1+\iota q}{2\pi c}}\left(\mathrm{B}\left(\frac{1}{2},1\right)\Phi_\iota^1(K_1-K_0) + (1+\iota q)\mathrm{B}\left(\frac{3}{2},1\right)K_0\Phi_\iota^2\right), \end{aligned} \tag{3.9}$$

where constants

$$\Phi_\iota^1 = \Phi\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{1+\iota q}{2},-d(1+\iota q)\right)$$

and

$$\Phi_\iota^2 = \Phi\left(\frac{3}{2},\frac{1}{2},\frac{5}{2};\frac{1+\iota q}{2},-d(1+\iota q)\right).$$

Since

$$\begin{aligned} \mathrm{P}^*\left(\overline{S}_t^{gen} \le u\right) =& \frac{\delta t}{\sqrt{2\pi}}\Bigg\{\int_0^\infty \Psi\left(\frac{\log u}{\sigma\sqrt{g}} + \frac{\sigma}{2}\sqrt{g}\right)g^{-\frac{3}{2}}\exp\left(-\frac{(\delta t)^2}{2g}\right)dg \\ & - e^{-\log u}\int_0^\infty \Psi\left(-\frac{\log u}{\sigma\sqrt{g}} + \frac{\sigma}{2}\sqrt{g}\right)g^{-\frac{3}{2}}\exp\left(-\frac{(\delta t)^2}{2g}\right)dg\Bigg\}, \end{aligned}$$

we get that

$$\begin{aligned} &\mathrm{P}^*\left(\overline{S}_t^{gen} \le u\right) \\ &= \frac{de^d}{\pi\sqrt{2}}\Bigg\{\mathrm{B}\left(\frac{1}{2},1\right)(K_1-K_0)\Big[(1+q)^{\frac{1}{2}}\Phi_+^1 - (1-q)^{\frac{1}{2}}e^{-\log u}\Phi_-^1\Big] \\ &\quad + \mathrm{B}\left(\frac{3}{2},1\right)K_0\Big[(1+q)^{\frac{3}{2}}\Phi_+^2 - (1-q)^{\frac{3}{2}}e^{-\log u}\Phi_-^2\Big]\Bigg\}. \end{aligned}$$

Proof of Corollary 2.2 Similar to (3.1), we have from (2.3) that

$$\mathrm{P}^*\left(\underline{S}_t^{gen} \le u\right) = \int_0^\infty \mathrm{P}\left(\max_{s\le g}\left(\frac{\sigma^2}{2}s + \sigma B_s\right) \ge -\log u\right)f(T^*(t),g)dg$$

and

$$
\begin{aligned}
&\mathrm{P}\left(\max_{s\le g}\left(\frac{\sigma^2}{2}s+\sigma B_s\right)\ge -\log u\right)\\
&=\Psi\left(\frac{\log u}{\sigma\sqrt{g}}+\frac{\sigma}{2}\sqrt{g}\right)+e^{\log u}\Psi\left(\frac{\log u}{\sigma\sqrt{g}}-\frac{\sigma}{2}\sqrt{g}\right)\\
&=\Psi\left(\frac{\log u}{\sigma\sqrt{g}}+\frac{\sigma}{2}\sqrt{g}\right)+e^{\log u}-e^{\log u}\Psi\left(-\frac{\log u}{\sigma\sqrt{g}}+\frac{\sigma}{2}\sqrt{g}\right). \qquad (3.10)
\end{aligned}
$$

Comparing (3.10) and (3.3), we obtain (2.9).

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